**Exam.Code:5003**
**Sub. Code: 11023**

**2125**

**NEP U.G. Common-Skill Enhancement Course**
**Third Semester**
**Mathematics**
**Paper: Mathematical Modeling**

**Time allowed: 3 Hours** **Max. Marks: 60**

***NOTE: Attempt <u>four</u> questions in all, including Question No. I which is compulsory and selecting one question from each Unit.***

x-x-x

1. (a) Describe the main classifications of mathematical models and explain how they guide the choice of a modelling framework. Briefly comment on the limitations of mathematical modelling and how these affect a model's practical use. **[2 Marks]**

   (b) Consider the growth of a savings account where a fixed amount is deposited each month. Describe how the variables and assumptions are chosen and explain how a basic mathematical model can be constructed to study the change in the account balance over time. **[2 Marks]**

   (c) A wooden artifact contains only 70% of the carbon fourteen found in fresh wood. Using the exponential decay law and a half life of 5568 years, estimate the age of the artifact and infer the likely age of the site. **[2 Marks]**

   (d) A small animal population grows at a rate proportional to its size. The population increases from 120 to 210 over four years. Using this information, formulate the exponential growth model, find the growth constant $k$, and estimate the population after six years. **[2 Marks]**

   (e) Consider the SIR model $dS/dt = -\beta SI$, $dI/dt = \beta SI - \gamma I$, and $dR/dt = \gamma I$, with total population $N = S + I + R$ constant. At the start of an outbreak $S(0) = 0.95N$, with $\beta = 0.3$ and $\gamma = 0.1$. Compute the effective reproduction number $R_{\text{eff}} = \beta S(0)/(\gamma N)$ and decide whether the infection grows or declines initially. **[2 Marks]**

   (f) For the competition model $\dfrac{dx}{dt} = rx\left(1 - \dfrac{x + \alpha y}{K}\right)$, explain how the parameter $\alpha$ measures the competitive effect of species $y$ on the logistic growth of species $x$, and describe how increasing $\alpha$ alters the growth of $x$. **[2 Marks]**

## UNIT - I

2. (a) Consider the differential equation $(3x^2y + 4y)\,dx + (x^3 + 2x)\,dy = 0$ *(i)* Determine whether the equation is exact *(ii)* If it is not exact, find an integrating factor that makes it exact *(iii)* Obtain the general solution. **[8 Marks]**

   (b) Consider the differential equation $\dfrac{dy}{dx} = \dfrac{x}{y} + \dfrac{y}{x}$. Use the substitution $y = vx$ to reduce the equation, solve for the general solution, and then find the particular solution satisfying $y(1) = 2$. **[8 Marks]**

3. (a) Solve the differential equation $(3x^2y^3e^y + y^3 + y^2)\,dx + (x^3y^3e^y - xy)\,dy = 0$. **[8 Marks]**

   (b) Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{y}{x + \sqrt{xy}}$ with $y(1) = 2$. Use a suitable substitution to convert the equation into a separable form and determine the particular solution that satisfies the given initial condition. **[8 Marks]**

## UNIT - II

4. (a) A fish population is assumed to follow the logistic model $dP/dt = 0.6P\,(1 - P/3000) - 200$, where $P(t)$ denotes the population at time $t$ in years. *(i)* Write the equilibrium condition by setting $dP/dt = 0$. *(ii)* Solve the resulting equation numerically and obtain the positive equilibrium population correct to the nearest whole number. *(iii)* Comment on whether the population survives or becomes extinct when the initial value is $P(0) = 1500$. **[8 Marks]**

   (b) A bacterial culture grows at a rate proportional to the number of bacteria present. The initial population is 500, and after two hours it has increased to 800. Using this information, formulate the differential equation governing the growth and explain the parameters you introduce. Determine the growth constant, derive the expression for the population $P(t)$, and estimate the population after five hours **[8 Marks]**

5. (a) Consider the discrete population model $P_{n+1} = rP_n(1 - P_n)$, where $0 < P_n < 1$ denotes the scaled population in generation $n$ and $r$ is a growth parameter. ($i$) Find the equilibrium values of the model. ($ii$) Analyse the stability of the positive equilibrium for $r = 2.4$. ($iii$) Briefly explain how the long term behaviour of the population changes as $r$ increases beyond 3 and approaches chaotic dynamics. **[8 Marks]**

   (b) A radioactive substance loses mass over time, and the decay rate is assumed to be proportional to the amount present. Using this assumption, derive the differential equation for the quantity $N(t)$ at time $t$ and explain the meaning of the parameters involved. Solve the equation with $N(0) = N_0$, define the half life, and obtain its relation to the decay constant. Conclude by interpreting the rate constant in terms of the residence time of a typical particle. **[8 Marks]**

## UNIT - III

6. (a) Consider the predator–prey system $dx/dt = 0.4x - 0.02xy$ and $dy/dt = -0.3y + 0.01xy$. Find the equilibrium points, determine their stability, and describe the qualitative behaviour of the populations near each equilibrium. **[8 Marks]**

   (b) In a simple SIS model the population is divided into susceptibles $S(t)$ and infectives $I(t)$ with total population constant. Infection occurs at rate $\beta$ and recovery at rate $\gamma$. Formulate the differential equations for $S(t)$ and $I(t)$, determine the equilibrium points, and state the condition for the existence of an endemic equilibrium. Briefly describe how the infection behaves depending on whether this condition is satisfied. **[8 Marks]**

7. (a) Consider the predator–prey system $dx/dt = 0.5\,x - 0.02\,x\,y$ and $dy/dt = -0.4\,y + 0.01\,x\,y$, where $x(t)$ and $y(t)$ denote the prey and predator populations. Determine the equilibrium points, compute the Jacobian at the nontrivial equilibrium and find its eigenvalues, classify the equilibrium, and use the linearisation to decide whether the solution with $x(0) = 40$ and $y(0) = 10$ approaches this equilibrium or shows oscillatory behaviour. **[8 Marks]**

   (b) In an SIR model the population is split into susceptibles $S(t)$, infectives $I(t)$, and recovered individuals $R(t)$, with total population fixed. Infection occurs at rate $\beta$ and recovery at rate $\gamma$. Formulate the differential equations for $S(t)$, $I(t)$, and $R(t)$, identify the disease free equilibrium, and state the condition under which an outbreak is possible. Briefly explain the role of the basic reproduction number in this model. **[8 Marks]**

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